TMS165/MSA350 Stochastic Calculus
Written exam Tuesday 26 October 2021 8.30–12.30

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Aids: Two sheets (=four pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed).

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated. Good Luck!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Can a non-constant random process have its variation process equal to its quadratic variation process? **(5 points)**

Task 2. Let X be a Poisson distributed random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and define $\mathcal{G} \subseteq \mathcal{F}$ by $\mathcal{G} = \{\emptyset, A, A^c, \Omega\}$ where $A = \{\omega \in \Omega : X(\omega) = 0\}$. Calculate $\mathbf{E}\{X|\mathcal{G}\}$. **(5 points)**

Task 3. Is there a function $f : \mathbb{R}\times[0,\infty) \to \mathbb{R}$ such that $\{\int_0^t B(r)\,dr + f(B(t),t)\}_{t\geq 0}$ is a martingale? **(5 points)**

Task 4. Solve the Stratonovich SDE $dX(t) = -\alpha\,dt + \sigma\,X(t)\,\partial B(t)$ for $t \geq 0$, $X(0) = x_0$, where α, σ and x_0 are real constants. **(5 points)**

Task 5. In the following list of probability measures for a random variable X, find all pairs of equivalent measures: (a) X is standard normal, (b) X is normal with expectation 1 and variance 2, (c) $\ln(X)$ is standard normal, (d) $\ln(X)$ is normal with expectation 1 and variance 2, (e) X is exponential distributed with expectation 1, (f) X is uniformly distributed over the interval $[0,1]$, (g) X is uniformly distributed over the interval $[0,2]$, (h) X is binomial distributed with parameters $n = 1$ and $p = 0.5$, (i) X is binomial distributed with parameters $n = 1$ and $p = 0.1$, (j) X is binomial distributed with parameters $n = 2$ and $p = 0.5$, (k) X is Poisson distributed with parameter $\lambda = 1$, (l) X is Poisson distributed with parameter $\lambda = 2$. **(5 points)**

Task 6. Grönwall's lemma that $\phi(t) \leq A + B\int_0^t \phi(s)\,ds$ implies $\phi(t) \leq A\,\mathrm{e}^{Bt}$ is true for constants $A, B \geq 0$ although we have only seen the proof for $A, B > 0$. How is the proof done for $A, B \geq 0$? **(5 points)**

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Task 1. Yes, this will be the case for any pure jump process with all jumps being of unit height.

Task 2. $\mathbf{E}\{X|\mathcal{G}\}(\omega) = 0$ for $\omega \in A$ and $\mathbf{E}\{X|\mathcal{G}\}(\omega) = \mathbf{E}\{X\}/\mathbf{P}\{X > 0\}$ for $\omega \in A^c$.

Task 3. Yes $f(x,t) = -t\,x$ will work as $\mathbf{E}\{\int_0^t B(r)\,dr - t\,B(t)|\mathcal{F}_s^B\} = \int_0^s B(r)\,dr + \mathbf{E}\{\int_s^t (B(r) - B(s))\,dr\} + (t-s)\,B(s) - t\,B(s)$ for $0 \leq s \leq t$ where the second expectation on the right-hand side is zero.

Task 4. By Theorem 5.20 in Klebaner's book the Stratonovicdh SDE is equivalent to the Itô SDE $dX(t) = (\frac{1}{2}\,\sigma^2\,X(t) - \alpha)\,dt + \sigma\,X(t)\,dB(t)$ for $t \geq 0$, $X(0) = x_0$. This in turn is a special case of the linear SDE treated in Section 5.3 with solution $X(t) = \mathrm{e}^{\sigma B(t)}\big(x_0 - \alpha\int_0^t \mathrm{e}^{-\sigma B(s)}\,ds\big)$.

Task 5. $\{$(a),(b)$\}$, $\{$(c),(d),(e)$\}$, $\{$(h),(i)$\}$ and $\{$(k),(l)$\}$.

Task 6. Add $\varepsilon > 0$ to A and B and use the proof we have done for these increased A and B constants. Then send $\varepsilon \downarrow 0$ afterwards.